# Lee Model on Riemannian Manifolds and Heat Kernel Methods

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• A simple renormalization problem, delta function potential in two dimensions.

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \sum_i g_i \delta(x - a_i). \qquad (1)$$

look for bound state solutions.

• there is a divergence!

• Work in the Fourier space, set $E = -\nu^2$,

$$\int [d^2k] e^{ik.x} [\frac{\hbar^2}{2m} k^2 \tilde{\psi}(k) + \nu^2 \tilde{\psi}(k) \qquad (2)$$

$$- \quad \sum_i g_i \psi(a_i) e^{-ik.a_i}] = 0.$$

$$\tilde{\psi}(k) = \sum_i g_i \psi(a_i) \frac{e^{-ik.a_i}}{\frac{\hbar^2}{2m}k^2 + \nu^2}, \qquad (3)$$

Find the wave function, inverse Fourier, and consistency equation:

$$\psi(a_i) = \sum_j g_j \psi(a_j) \int [d^2k] \frac{e^{ik.(a_i - a_j)}}{\frac{\hbar^2}{2m}k^2 + \nu^2}. \qquad (4)$$

Regroup these terms,

$$[g_i^{-1}(\Lambda) - \int_{|k|<\Lambda}[d^2k]\frac{1}{\frac{\hbar^2}{2m}k^2+\nu^2}]\psi(a_i) \quad (5)$$

$$- \sum_{j,j\neq i}\frac{g_j}{g_i}\psi(a_j)\int_{|k|<\Lambda}[d^2k]\frac{e^{ik.(a_i-a_j)}}{\frac{\hbar^2}{2m}k^2+\nu^2} = 0.$$

- Choose the coupling constants as

$$\frac{1}{g_i(\Lambda)} = \int_{|k|<\Lambda}[d^2k]\frac{1}{\frac{\hbar^2}{2m}k^2+\mu_i^2}. \quad (6)$$

We find a matrix equation,

$$\Phi_{ij}(-\nu^2)\psi(a_j) = 0 \quad (7)$$

where

$$\Phi_{ij}(-\nu^2) = \frac{m}{\pi\hbar^2}\begin{cases}\ln(\frac{\nu}{\mu_i}) & i=j \\ -K_0(\frac{\sqrt{2m}}{\hbar}\nu|a_i-a_j|) & i\neq j\end{cases}. (8)$$

- Two center case.

$$\det\begin{pmatrix}\ln(\frac{\nu}{\mu_1}) & -K_0(\frac{\sqrt{2m}}{\hbar}\nu|a_1-a_2|) \\ -K_0(\frac{\sqrt{2m}}{\hbar}\nu|a_2-a_1|) & \ln(\frac{\nu}{\mu_2})\end{pmatrix} = 0$$

$$\begin{aligned} K_0(x) &= \int_0^\infty dt e^{-x\cosh t} \\ K_0(x) &\approx -\ln(\frac{x}{2}) \quad x\to 0 \end{aligned}$$

$$K_0(x) \quad \approx \quad \sqrt{\frac{\pi}{2x}}e^{-x} \quad x \to \infty \tag{9}$$

$\ln^2(x)$ is increasing starting from 0 to $\infty$ as $x \to \infty$, $K_0^2(x)$ is monotonically decreasing starting from $\infty$ to 0 exponentially there is always one intersection. Small $x$ analysis reveals that there are two intersections if

$$|a_1 - a_2| > \frac{\sqrt{2}\hbar e^\gamma}{\sqrt{\mu_1\mu_2 m}}. \tag{10}$$

• Ordinary eigenvalue equation is replaced by a nonlinear matrix equation! (J. Hoppe MIT thesis 1980)

♯ (B. Altunkaynak and F. Erman) Abstract setting:

$$H = H_0 - \sum_i g_i|f_i >< f_i|, \tag{11}$$

Find the Greens function which is the inverse of this operator $(H - z)^{-1}$.

- algebraic solution:

$$(H_0 - z)|\psi > - \sum_i g_i |f_i >< f_i|\psi >= |\chi >,$$

$$|\psi >= (H_0 - z)^{-1}|\chi >$$
$$- \sum_i g_i (H_0 - z)^{-1}|f_i >< f_i|\psi >,$$

solve for the unknown,

$$< f_j|\psi >=< f_j|(H_0 - z)^{-1}|\chi >$$
$$- \sum_i g_i < f_j|(H_0 - z)^{-1}|f_i >< f_i|\psi >,$$

$$[\frac{1}{g_j} - < f_j|(H_0 - z)^{-1}|f_j >] < f_j|\psi >$$
$$+ \sum_{i \neq j} < f_j|(H_0 - z)^{-1}|f_i >< f_i|\psi >$$
$$=< f_j|(H_0 - z)^{-1}|\chi > .$$

$$(H-z)^{-1} = (H_0 - z)^{-1}$$

$$+(H_0 - z)^{-1} \sum_{i,j} |f_i > \Phi_{ij}^{-1}(z) < f_j|(H_0 - z)^{-1}, \quad (12)$$

where

$$\Phi_{ij}(z) = \begin{cases} g_i^{-1} - < f_i|(H_0 - z)^{-1}|f_j > & i = j \\ \frac{g_j}{g_i} < f_i|(H_0 - z)^{-1}|f_j > & i \neq j \end{cases} . \quad (13)$$

• the bound state solutions are coming from the poles of the resolvent below the starting point of the branch cut of the free part for the non-compact case or below the poles of the free part in the compact case.

• This allows us to pass to Riemannian Manifolds! Choose $|f_i >$ to be Gaussian bump functions centered around some points $a_i$, approaching delta functions. A natural choice for a manifold $f_i^{\epsilon}(x) = K_{\epsilon}(a_i, x)$.

Laplace-Beltrami operator:

$$\nabla_g^2 = -\frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j), \tag{14}$$

Introduce the Heat kernel,

$$K_t(x,y) = < x| e^{-\frac{t}{\hbar}(\frac{\hbar^2}{2m} \triangle_g)} |y> \tag{15}$$

Solves the Heat equation (Euclidean Schrodinger Eqn).

$$\begin{aligned} K_t(x,y) &= K_t(y,x) \quad , \\ \frac{\partial K_t(x,x')}{\partial t} - \nabla_g^2 K_t(x,x') &= 0 \\ \lim_{t \to 0^+} K_t(x,y) &= \delta_g(x,y), \end{aligned}$$

$$\begin{aligned} \int_{\mathcal{M}} d_g x \; K_{t_1}(x,z) K_{t_2}(z,y) &= K_{t_1+t_2}(x,y) \\ K_t(x,y) &= \sum_\lambda e^{-\lambda t} f_\lambda(x) f_\lambda(y) \end{aligned}$$

Here,

$$\begin{aligned} \Delta_g f_\lambda &= \lambda f_\lambda \\ 0 &\le \lambda_1 \le \lambda_2 \le \lambda_3 \le ..... \to \infty \end{aligned}$$

For stochastically complete manifolds(for example compact manifolds):

$$\int_{\mathcal{M}} d_g x \; K_t(x,y) = 1 \tag{16}$$

This means that the total heat content is preserved.

• Theorem: Given a geodesically complete manifold assume that for some point $x$ we have

$$\int_R^\infty \frac{r dr}{\ln V(x,r)} = \infty, \tag{17}$$

then the stochastic completeness holds.

On a product manifold $\mathcal{M}_1 \times \mathcal{M}_2$,

$$K_t(x,y) = K_t^{(1)}(x_1,y_1) K_t^{(2)}(x_2,y_2) \tag{18}$$

Decay of the heat kernel, $\mathcal{M}$ noncompact:

$$K_t(x,y) \to 0 \text{ as } t \to \infty \tag{19}$$

$\mathcal{M}$ compact:

$$K_t(x,y) \to \frac{1}{V(\mathcal{M})} \text{ as } t \to \infty. \tag{20}$$

• Free Greens function for Re($z$) $< 0$,

$$(H_0 - z)^{-1} = \frac{1}{\hbar}\int_0^\infty e^{-\frac{t}{\hbar}(\frac{\hbar^2}{2m}\triangle_g - z)} dt, \qquad (21)$$

should be continued analytically to its largest set in the entire complex plane.

In general, it is possible to write for a positive operator $H_0$ a heat kernel,

$$\langle a_i| (H_0 - z)^{-1} |a_j\rangle = \frac{1}{\hbar}\int_0^\infty e^{\frac{zt}{\hbar}} K_t(a_i, a_j)\, dt\ , \qquad (22)$$

Heat kernel for $\mathbf{R}^n$:

$$\begin{aligned} K_t(x,y) &= \int [d^n k] e^{ik.(x-y)} e^{-k^2 t} \\ &= \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} \end{aligned}$$

Green's function for $\mathbf{R}^n$:

$$R_0(x,y|z) = \int [d^n k] \frac{e^{ik.(x-y)}}{\hbar^2 k^2/2m - z} \qquad (23)$$

$$\Phi^\epsilon_{ij}(z) = \begin{cases} g_i^{-1}(\epsilon) - \int d_g x d_g y K_{\epsilon/2}(a_i, x) R_0(x,y|z) K_{\epsilon/2}(y, a_i) \\ -\frac{g_j(\epsilon)}{g_i(\epsilon)} \int d_g x d_g y K_{\epsilon/2}(a_i, x) R_0(x,y|z) K_{\epsilon/2}(y, a_j) \end{cases}$$

Let us look at the diagonal term,

$$\begin{aligned} g_i^{-1}(\epsilon) \;\; - \;\; & \int_0^\infty \frac{dt}{\hbar} \int d_g x d_g y \\ & K_{\epsilon/2}(a_i, x) K_t(x, y) e^{tz/\hbar} K_{\epsilon/2}(y, a_i). \end{aligned}$$

Reproducing property gives us

$$g_i^{-1}(\epsilon) - \int_0^\infty \frac{dt}{\hbar} K_{t+\epsilon}(a_i, a_i) e^{tz/\hbar} \tag{24}$$

Shift of integration variable shows that

$$g_i^{-1}(\epsilon) - \int_\epsilon^\infty \frac{dt}{\hbar} K_t(a_i, a_i) e^{tz/\hbar} \tag{25}$$

• as $t \to 0^+$, we have the asymptotic expansion,

$$K_t(x, x) \sim \left(4\pi \frac{\hbar t}{2m}\right)^{-1} \sum_{k=0}^{\infty} u_k(x, x) \left(\frac{\hbar t}{2m}\right)^k , \tag{26}$$

• The coefficients $u_k(x, x)$ are universal polynomials in the curvature tensor and its covariant derivatives. e.g.

$$\begin{aligned} u_0(x, x) &= 1 \\ u_1(x, x) &= \frac{1}{6} R \\ u_2(x, x) &= \frac{1}{360} (2R_{ijkl} R^{ijkl} + 2R_{jk} R^{jk} + 5R^2 \\ & \quad -12 \Delta_g R) \end{aligned}$$

Choose,

$$g_i^{-1}(\epsilon, \mu_i) = \frac{1}{\hbar} \int_\epsilon^\infty e^{\frac{-\mu_i^2 t}{\hbar}} K_t(a_i, a_i)\, dt\ . \qquad (27)$$

• this gives us

$$\Phi_{ij}(z) = \begin{cases} \frac{1}{\hbar} \int_0^\infty K_t(a_i, a_i) \left[ e^{\frac{-\mu_i^2 t}{\hbar}} - e^{\frac{zt}{\hbar}} \right] dt & i = j \\ -\frac{1}{\hbar} \int_0^\infty e^{\frac{zt}{\hbar}} K_t(a_i, a_j)\, dt & i \neq j \end{cases} \qquad (28)$$

$$\begin{aligned} R(x, y|z) &= R_0(x, y|z) \\ &+ \sum_{i,j=1}^{N} R_0(x, a_i|z)\, \Phi_{ij}(z)^{-1} R_0(a_j, y|z)\ . \end{aligned}$$

• All the information contained in the resolvent: How do we find the bound states? Density of states? Meta principle:

$$R(x, y|z) = \sum_{\lambda \in \sigma_d(H)} \frac{\psi_\lambda^*(x) \quad \overbrace{\psi_\lambda(y)}^{\text{normalizable}}}{\lambda - z}$$

$$+ \int_{\lambda \in \sigma_c(H)} d\lambda \, \frac{\rho(\lambda)}{\lambda - z} \zeta^*_\lambda(x) \underbrace{\zeta_\lambda(y)}_{\text{generalized}} .$$

♯ For bound states:

$$\psi^*_n(x)\psi_n(y) = \frac{1}{2\pi i} \oint_{\Gamma_n \ni \lambda_n} dz R(x, y|z) \tag{29}$$

$$\begin{aligned} \psi_n(x) \;&=\; \left[ \int_0^\infty dtte^{-\nu_n^2 t} \sum_{i,j} K_t(a_i, a_j) A^*_i(\nu_n) A_j(\nu_n) \right]^{-\frac{1}{2}} \\ &\times \; \int_0^\infty e^{-\frac{t\nu_n^2}{\hbar}} \sum_{i=1}^{N} A_i(\nu_n) K_t(a_i, x) \frac{dt}{\hbar} \, , \end{aligned} \tag{30}$$

♯ how do we know that the spectrum is bounded from below? Resolvent!

• Gershgorin theorem: eigenvalues of a matrix $A = [a_{ij}]$ all lie in the discs:

$$\bigcup_{i=1}^{n} \{\lambda \in \mathbf{C} : \; |\lambda - a_{ii}| \leq \sum_{i \neq j} |a_{ij}|\} \tag{31}$$

We want $\Sigma(\nu) = 0$ not to be an eigenvalue!

$$|\Phi_{ii}(-\nu^2)| > \sum_{i \neq j} |\Phi_{ij}(-\nu^2)| \text{ for all } i \tag{32}$$

Is it possible to find a finite value of $\nu$ satisfying this condition? General observation:

$$\Phi_{ii}(-\nu^2) \approx \ln\frac{\nu}{\mu_i}, \text{ as } \nu \to \infty,$$
$$\frac{\partial|\Phi_{ii}(-\nu^2)|}{\partial\nu} = \frac{2\nu}{\hbar^2}\int_0^\infty tK_t(a_i,a_i)e^{-(\frac{\nu^2 t}{\hbar})}dt > 0$$

and

$$|\Phi_{ij}(-\nu^2)| \approx e^{-\nu d(a_i,a_j)} \text{ as } \nu \to \infty$$
$$\frac{\partial|\Phi_{ij}(-\nu^2)|}{\partial\nu} = -\frac{2\nu}{\hbar^2}\int_0^\infty tdtK_t(a_i,a_j)e^{-(\frac{\nu^2 t}{\hbar})} < 0.$$

As we increase $\nu$ the inequality will be satisfied for some value! Thus $E_{gr} > -(\nu^*)^2$, energy is bounded from below!

- Examples: $S^2$

$$\Phi_{ij}(z) = \frac{m}{2\pi\hbar^2}\begin{cases} \phi(\frac{\mu_i}{\mu_R}) - \phi(\frac{\sqrt{-z}}{\mu_R}) & i = j \\ -\sum_{l\geq 0}\frac{2l+1}{l(l+1)-z/\mu_R^2}P_l(1-\frac{d_{ij}^2}{2}) & i \neq j \end{cases} \tag{33}$$

$\mu_R^2 = \frac{\hbar^2}{2mR^2}$. The function $\phi$ defined as

$$\phi(z) = \frac{1}{z^2} - H_{\frac{1}{2}-\sqrt{\frac{1}{4}-z^2}} - H_{\frac{1}{2}+\sqrt{\frac{1}{4}-z^2}}, \tag{34}$$

here $H_z = \psi(z+1) + \gamma$, where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$

- $H^2$ Case: Upper half-plane model, defined by the metric

$$\cosh \frac{d(x,y)}{R} = 1 + \frac{|x-y|^2}{2\, x_2\, y_2}\,, \tag{35}$$

$$\begin{aligned} K_t(x,y) &= \frac{\sqrt{2}}{(4\pi \left[\frac{\hbar}{2mR^2}\right] t)^{3/2}} \frac{e^{-\frac{\hbar}{2mR^2}\frac{t}{4}}}{R^2} \\ &\quad \int_{\frac{d(x,y)}{R}}^{\infty} \frac{r e^{-\frac{r^2}{4}\frac{2mR^2}{\hbar}\frac{1}{t}}}{\sqrt{\cosh r - \cosh \frac{d(x,y)}{R}}}\, dr \end{aligned}$$

$$\Phi_{ii}(z) = \frac{m\sqrt{2}}{2\pi\hbar^2}\left[\psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{z}{\mu_R^2}}\right) - \psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu_i^2}{\mu_R^2}}\right)\right] \tag{36}$$

$$\begin{aligned} \Phi_{ij}(z) &= -\frac{m}{2\pi\hbar^2}\int_{\frac{d_{ij}}{R}}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh \frac{d_{ij}}{R}}}\, dr, \\ &= \frac{\sqrt{2}}{4\pi R^2 \mu_R^2} Q_{1/2\sqrt{1-4z/\mu_R^2}-1/2}(\cosh \frac{d(a_i,a_j)}{R}) \end{aligned}$$

Asymptotics,

$$\psi(z) \approx \ln z \text{ as } z \to \infty, \quad Q_\nu(x) \approx \frac{e^{-\nu x}}{\sqrt{\nu}} \text{ as } \nu \to \infty. \tag{37}$$

- a comparison theorem.

$$\Sigma^m(\nu) = (A^m(\nu), \Phi(-\nu^2)A^m(\nu)). \tag{38}$$

By the Feynmann-Helmann theorem,

$$\begin{aligned} \frac{\partial \Sigma^m(\nu)}{\partial \nu} &= (A^m(\nu), \frac{\partial \Phi(-\nu^2)}{\partial \nu} A^m(\nu)) \\ \frac{\partial \Phi_{ij}(-\nu^2)}{\partial \nu} &= \int_0^\infty dt (2\nu t) K_t(a_i, a_j) e^{-\nu^2 t} \end{aligned}$$

the integral is finite in two and three dimensions due to the short time behaviour of the heat kernel.

$$\frac{\partial \Sigma^m(\nu)}{\partial \nu} > 0. \tag{39}$$

- Graphical interpretation spectral flow!

$$\Phi_{(N+1)\times(N+1)}(-\nu^2) = \left( \begin{array}{ccc|c} & & & \bullet \\ & & & \bullet \\ & & & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet \end{array} \right) \tag{40}$$

$$\Sigma_1(\nu) < \tilde{\Sigma}_1(\nu) < \Sigma_2(\nu) < ... < \tilde{\Sigma}_N(\nu) < \Sigma_{N+1}(\nu). \tag{41}$$

$\Sigma_k(\tilde{\nu}_k^*) < \tilde{\Sigma}_k(\tilde{\nu}_k^*) = 0 < \Sigma_{k+1}(\tilde{\nu}_k^*)$, $\Sigma_k(\nu_k^*) = 0$ for $\nu_k^* > \tilde{\nu}_k^*$.

$$E_k = -\nu_k^2 < \tilde{E}_k = -\tilde{\nu}_k^2. \tag{42}$$

• global bounds on the heat kernel (Li-Yau) for nonnegative Ricci curvature.

$$\begin{aligned} K_t(x,y) &\leq \frac{C(\varepsilon)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}\, e^{-\frac{d(x,y)^2}{(4+\varepsilon)t}} . \\ K_t(x,y) &\geq \frac{c(\varepsilon)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}\, e^{-\frac{d(x,y)^2}{(4-\varepsilon)t}} . \end{aligned} \tag{43}$$

$V(x,s)$ volume of the geodesic sphere of radius $s$ arond $x$. Special bounds, for example Cartan-Hadamard manifolds(see below).

• How does the volume grow? For small $r$,

$$V_n(x,r) = r^n V_n(1)[1 - \frac{R(x)}{6(n+2)} r^2 + ...] \tag{44}$$

• (**Bishop-Gunther**)Global estimates: a complete $n$-dim Riemannian manifold, $V(x,r)$ the volume of a ball which does not meet the cut locus of $x$. $V^k(r)$ volume of a ball of radius $r$ in the complete simply connected Riemannian manifold with constant curvature k,

1) if $Ric(U,V) \geq (n-1)ag(U,V)$,

$$V(x,r) \leq V^a(r) \tag{45}$$

2)if $K \leq b$

$$V(x,r) \geq V^b(r). \tag{46}$$

• (**Coulhon-Grigoryan**) On a geodesically complete Riemannian manifold assume that for some point $x$ and all $r > 0$,

$$\begin{aligned} V(x,2r) &\leq CV(x,r) \text{ and for all t} > 0 \\ K_t(x,x) &\leq \frac{C}{V(x,\sqrt{t})} \text{ then, for all t} > 0 \\ K_t(x,x) &\geq \frac{c(C)}{V(x,\sqrt{t})}. \end{aligned} \tag{47}$$

• question: when do we have upper bounds of the form,

$$K_t(x,x) \leq \frac{C}{f(t)} \quad ? \tag{48}$$

(see the lectures by Grigoryan)

• **Theorem (Coulhon-Grigoryan)** Let $\mathcal{M}$ be a geodesically complete Riemannian manifold. Assume that for some point $x \in \mathcal{M}$ and all $r > r_0$,

$$V(x, r) \leq Cr^N \tag{49}$$

with some constants $C$ and $N$. Then for all $t > t_0$;

$$K_t(x, y) \geq \frac{1}{4V(x, K\sqrt{t \ln t})} \tag{50}$$

where $K > 0$ depends on $x, r_0, C, N$ and $t_0 = max(r_0^2, e)$.

**Faber-Krahn inequality**:

$$\lambda_1(\Omega) \geq cV(\Omega)^{-2/n}, \tag{51}$$

where $\Omega$ is a closed and bounded domain in $\mathbf{R}^n$ and $\lambda_1(\Omega)$ first eigenvalue of the Laplace operator. generalized FK:

$$\lambda_1(\Omega) \geq \Lambda[V(\Omega)], \tag{52}$$

where $\Lambda$ is a decreasing function on $(0, \infty)$, $\Omega$ is a precompact region in $\mathcal{M}$.

• Theorem: Assume that the manifold admits a FK function $\Lambda$, define $f(t)$ by

$$t = \int_0^{f(t)} \frac{dv}{v\Lambda(v)}. \tag{53}$$

Then for all $t > 0$, $x \in \mathcal{M}$ and $\epsilon > 0$, we have

$$K_t(x,x) \leq \frac{2\epsilon^{-1}}{f((1-\epsilon)t)}. \tag{54}$$

Example: $\Lambda(v) \succ \begin{cases} v^{-2/n} \text{ for } t \leq 1 \\ v^{-2/m} \text{ for } t > 1 \end{cases}$ then

$$K_t(x,x) \leq \frac{c}{t^{m/2}} \text{ for } t > 1. \tag{55}$$

• A manifold $\mathcal{M}$ admits the isoperimetric function $I$ if for any precompact open set $\Omega \in \mathcal{M}$ with smooth boundary

$$\sigma(\partial\Omega) \geq I(V(\Omega)). \tag{56}$$

♭ $\mathbf{R}^n$ admits the isoperimetric function $\omega_n^{1/n} n^{1-1/n}$.

$\sharp$ Theorem: Let $I(v)$ be a nonnegative continuous function on $\mathbf{R}_+$ such that $I(v)/v$ is nonincreasing. Assume that $\mathcal{M}$ admits the isopermteric fuction $I$. Then $\mathcal{M}$ admits the FK function

$$\Lambda(v) = \frac{1}{4}(\frac{I(v)}{v})^2. \tag{57}$$

• If we have the on diagonal estimate for all $x \in \mathcal{M}$,

$$K_t(x,x) \leq \frac{c}{f(t)}, \tag{58}$$

where $f$ is an increasing positive function, satisfying the regularity condition $(*)$ below, then for all $t > 0$ and $D > 2$ and for some $\epsilon > 0$ we have

$$K_t(x,y) \leq \frac{C}{f(\epsilon t)} exp(-\frac{d^2(x,y)}{2Dt}). \tag{59}$$

$(*)$ Regularity: There are numbers $A \geq 1$ and $a > 1$ such that

$$\frac{f(as)}{f(s)} \leq A\frac{f(at)}{f(t)} \text{ for all } 0 < s < t. \tag{60}$$

$\sharp$ example: If $A > 1$ and

$$f(2t) \leq Af(t), \tag{61}$$

then $(*)$ holds with $a = 2$.

• **Remark:**A general bound for the gorund state energy for the multi-delta functions could be proved using the upper and lower bounds given above.

• **Remark:** How do we know that the above formulae define actually a quantum Hamiltonian $H_R$ after the renormalization?

We have a formula for the resolvent, how do we know that this is the resolvent of a Hamiltonian, i. e., there is a closed densely defined self-adjoint operator $H_R$ such that,

$$R(z) = \frac{1}{H_R - z} \tag{62}$$

This is the case iff the resolvent family satisfies the resolvent equation on an unbounded region $\Delta$ of the complex plane,

$$R(z_1)R(z_2) = (z_1 - z_2)^{-1}(R(z_1) - R(z_2)) \tag{63}$$

and there is a sequence $\zeta_n \in \Delta$ such that $|\zeta_n| \to \infty$ and

$$||[\zeta_n R(\zeta_n) + 1]\psi|| \to 0 \quad \text{as } n \to \infty \tag{64}$$

for all $\psi \in \mathcal{H}$. Using the above set of inequalities we can prove that there is a quantum Hamiltonian $H_R$. (an alternative is to use the Kato's theorem of Resovent convergence).

• (with F. Erman) we introduce the nonrelativistic Lee model

$$H_\epsilon = H_0 + H_{I,\epsilon} \; , \tag{65}$$

$$H_0 = \int_{\mathcal{M}} d_g x \; \phi^\dagger(x) \left( -\frac{1}{2m} \nabla_g^2 + m \right) \phi(x) \; , \tag{66}$$

$$\begin{aligned} H_{I,\epsilon} &= \mu(\epsilon) \frac{1-\sigma_3}{2} \\ &+ \lambda \int_{\mathcal{M}} d_g x \; \rho_\epsilon(x,a) \left( \phi(x)\, \sigma_- + \phi^\dagger(x)\, \sigma_+ \right) \end{aligned}$$

♯ there is a conserved charge

$$Q = -\frac{1-\sigma_3}{2} + \int_{\mathcal{M}} d_g x \; \phi^\dagger(x) \phi(x) \tag{67}$$

• the Hilbert space of the theory is $\mathcal{F}_B(\mathcal{H}) \otimes \mathbf{C}^2$. Regularization of this theory is best done by the heat kernel. In fact the natural choice for $\rho_\epsilon(x,y)$ is the heat kernel.

$$< x | (-\frac{\nabla_g^2}{2m} + m - E)^{-1} | y > = \int_0^\infty dt K_{t/2m}(x,y) e^{-(m-E)t} . \tag{68}$$

$$\begin{pmatrix} H_0 - E & \lambda \int_{\mathcal{M}} d_g x \; K_\epsilon(x,a)\, \phi^\dagger(x) \\ \lambda \int_{\mathcal{M}} d_g x \; K_\epsilon(x,a)\, \phi(x) & H_0 - E + \mu(\epsilon) \end{pmatrix} . \tag{69}$$

Define a Green's function on the Fock space, using a formula of Rajeev:

$$R_\epsilon(E) = \frac{1}{H_\epsilon - E} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} , \qquad (70)$$

where

$$\begin{aligned} \alpha &= \frac{1}{H_0 - E} + \frac{1}{H_0 - E}\, b^\dagger\, \Phi_\epsilon^{-1}(E)\, b\, \frac{1}{H_0 - E} \\ \beta &= -\Phi_\epsilon^{-1}(E)\, b\, \frac{1}{H_0 - E} \\ \delta &= \Phi_\epsilon^{-1}(E) \\ b &= \lambda \int_{\mathcal{M}} d_g x\, K_\epsilon(x,a)\, \phi(x) \,. \end{aligned} \qquad (71)$$

$\Phi_\epsilon(E)$ is called the principal operator

$$\begin{aligned} \Phi_\epsilon(E) \;&=\; H_0 - E + \mu(\epsilon) \\ &\quad -\lambda^2 \textstyle\int_{\mathcal{M}} d_g x\, d_g y\; K_\epsilon(x,a) K_\epsilon(y,a)\; \phi(x) \displaystyle\frac{1}{H_0 - E} \phi^\dagger(y) \,, \end{aligned}$$

$$\begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} . \qquad (72)$$

$$\begin{aligned} \alpha &= a^{-1} + a^{-1} b^\dagger [d - b a^{-1} b^\dagger]^{-1} b a^{-1}, \\ \beta^\dagger &= -a^{-1} b^\dagger (d - b a^{-1} b^\dagger)^{-1} \\ \delta &= (d - b a^{-1} b^\dagger)^{-1} \end{aligned}$$

We need a commutation relation for the normal ordering:

$$\frac{1}{H_0 - E}\phi^\dagger(x) = \int_{\mathcal{M}} d_g x' \, \phi^\dagger(x') \int_0^\infty ds \, e^{-s(H_0 - E + m)} \, K_{s/2m}(x, x')$$

$$\begin{aligned}\Phi_\epsilon(E) &= H_0 - E - \lambda^2 \int_{\epsilon/2}^\infty ds \int_{\mathcal{M}} d_g x \, d_g y \; K_{s/2m}(x, a) \\ &\times \; K_{s/2m}(y, a)\phi^\dagger(x) \, e^{-(s-\epsilon/2)(H_0 + 2m - E)} \, \phi(y) \\ +\mu(\epsilon) &- \lambda^2 \int_\epsilon^\infty ds K_{s/2m}(a, a) \, e^{-(s-\epsilon)(H_0 + m - E)} \; .\end{aligned}$$

The last term is divergent! We can cure this by choosing $\mu(\epsilon)$ as

$$\mu(\epsilon) = \mu + \lambda^2 \int_\epsilon^\infty ds K_{s/2m}(a, a) e^{-s(m-\mu)} \qquad (73)$$

As a result we find the principal operator:

$$\begin{aligned}&\Phi(E) = H_0 - E + \mu - \lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x \, d_g y \\ &K_{s/2m}(x, a) K_{s/2m}(y, a) \phi^\dagger(x) \, e^{-s(H_0 + 2m - E)} \, \phi(y) \\ + \; &\lambda^2 \int_0^\infty ds K_{s/2m}(a, a) [e^{-s(m-\mu)} - e^{-s(H_0 + m - E)}] \; .\end{aligned}$$

• The bound states are solutions to the equation

$$\Phi(E)|\Psi >= 0 \tag{74}$$

• rigorous estimates on the ground state energy

$$\Phi(E) = K(E) - U(E) \ , \tag{75}$$

$$K(E) = H_0 - E + \mu \ , \tag{76}$$

$$\tilde{U}'(E) = K(E)^{-1/2}\, U'(E)\, K(E)^{-1/2},$$

$$\Phi(E) > K(E)^{1/2}(1 - \tilde{U}'(E))K(E)^{1/2}. \tag{77}$$

If $||\tilde{U}'(E)|| < 1$ , the operator is positive and hence no zeros appear. It can be shown that,

$$||\tilde{U}'(E)|| \ < \ n\, \frac{\lambda^2\, \Gamma(2)}{\Gamma(1/2)^2} \int_0^\infty s\, ds\; e^{-s(nm+\mu-E)} \int_0^1 \frac{du_1\, du_2\, du_3}{(u_1\, u_2)^{1/2}} \delta(u_1 + u_2 + u_3 - 1) \left[K_{s(u_1+u_3)/m}(a,a)\right]^{1/2} \left[K_{s(u_2+u_3)/m}(a,a)\right]^{1/2}$$

Cartan-Hadamard manifolds, which are geodesically complete simply connected non-compact Riemannian manifolds with non-positive sectional curvature bounded from above by $-K^2$

[• **Cartan-Hadamard** Theorem: A complete, simply connected Riemannian manifold with non-positive sectional curvature is diffeomorphic to $\mathbf{R}^n$. (these are called the Cartan-Hadamard manifolds)]

$$K_{s/2m}(x,x) \leq \frac{C}{(s/2m)^{3/2}}\,, \tag{78}$$

Closed compact manifolds with (**Wang**) Ricci curvature bounded from below by $-\kappa$.

$$K_{s/2m}(a,a) \leq \frac{1}{V(\mathcal{M})} + C(\kappa, V(\mathcal{M})))(s/2m)^{-3/2}\,, \tag{79}$$

$$E_{gr} \geq nm + \mu - n^2\lambda^4 F^2 \tag{80}$$

• Mean Field Approximation

$$\begin{aligned}\phi(E,u) &= nh_0(u) - E + \mu + \lambda^2 \int_0^\infty ds\, K_{s/2m}(a,a) \\ &\times \left[e^{-s(m-\mu)} - e^{-s(nh_0(u)+m-E)}\right] \\ &- n\lambda^2 \int_0^\infty ds \int_{\mathcal{M}} d_g x\, d_g y\, K_{s/2m}(x,a)\, K_{s/2m}(y,a)\end{aligned}$$

$$u^*(x)\, e^{-s(nh_0(u)+2m-E)}\, u(y) \ , \qquad (81)$$

called principal function and we have defined

$$h_0(u) = \int_{\mathcal{M}} d_g x \, \left( \frac{|\nabla_g u(x)|^2}{2m} + m|u(x)|^2 \right) \ . \quad (82)$$

We should solve the equation $\phi(E, u) = 0$

$$\begin{aligned} \chi &= nh_0(u) - E \\ v(x) &= [2m(2m+\chi)]^{-3/4} u(x) \ , \end{aligned} \qquad (83)$$

new wave functions $v(x)$ are normalized with respect to the new metric $\tilde{g}_{ij} = [2m(2m+\chi)] g_{ij}$

$$\int_{\mathcal{M}} d_{\tilde{g}} x \ |v(x)|^2 = \int_{\mathcal{M}} d_g x \ |u(x)|^2 = 1 \ . \qquad (84)$$

a dimensionless parameter $s' = (2m+\chi)s$,

$$\begin{aligned} & (2m+\chi)^{-1/2} \Big( \chi + \mu \\ + \ & \lambda^2 \int_0^\infty ds \ K_{s/2m}(a,a) \left[ e^{-s(m-\mu)} - e^{-s(\chi+m)} \right] \Big) \\ = \ & \underbrace{n\lambda^2 (2m)^{3/2} \int_0^\infty ds' \ \left| \int_{\mathcal{M}} d_{\tilde{g}} x \ K_{s'}(x,a;\tilde{g})\, v(x) \right|^2 e^{-s'}}_{nU} \end{aligned}$$

the inverse function of the left hand side $f_1(nU)$,

$$\chi = f_1(nU) \; , \tag{85}$$

$$E = nm + 2mnK[v] + (nK[v] - 1) f_1(nU) \; , \tag{86}$$

Interesting case if $nK[v] < 1$, compact manifolds.

$$\chi + \mu \le \frac{n\lambda^2}{\chi + 2m} \Big[ \frac{1}{V} + 2m \int_0^\infty ds (1 - e^{-s(\chi+2m)/2})$$
$$\times \; \Big( K_{s/2m}(a,a;g) - \lim_{s\to\infty} K_{s/2m}(a,a;g) \Big) \Big]$$
$$\times \left[ 1 + \frac{(\chi+2m)}{2m} K[v] \right] \; .$$

the upper bound estimate of the heat kernel for closed compact manifolds with Ricci curvature bounded from below by $-\kappa$, taking the integral with respect to $s$, we find that

$$\begin{aligned} \chi + \mu \;\; & \le \;\; \frac{n\lambda^2}{\chi + 2m} \left[ 1 + \left( \frac{\chi + 2m}{2m} \right) K[v] \right] \\ & \times \;\; \Big[ \frac{1}{V(\mathcal{M})} + \sqrt{2\pi} (2m)^{5/2} A (\chi + 2m)^{1/2} \Big] \end{aligned}$$

For this to be true,

$$\begin{aligned} E_{gr} \;\; & \ge \;\; nm + 2mnK[v] \\ & - \;\; (1 - nK[v]) \left( B_1 n^{2/3} + B_2 n^{1/3} + B_3 + \cdots \right) \end{aligned}$$

• Thus the energy per particle is finite as $n \to \infty$.

• One can go through the same thing for the relativistic bosons(with B. Kaynak):

$$\begin{aligned} H_0 &= \int_{\mathcal{M}} d_g x : \phi^\dagger(x)(-\nabla_g^2 + m^2)\phi(x) : \\ H_{I,\epsilon} &= \mu(\epsilon)\frac{1-\sigma_3}{2} \\ &+ \lambda(\epsilon)\int_{\mathcal{M}} d_g x K_\epsilon(a,x) \\ &\qquad (\phi^{(+)}(x)\sigma_+ + \phi^{(-)}(x)\sigma_-), \end{aligned}$$

where : : denotes the normal ordering and $\phi^{(+)}$ refers to the annihilation operator part and similarly $\phi^{(-)}$ refers to the creation operators.

we need the subordination identity for positive operators:

$$e^{-sA} = \frac{\sqrt{\pi}}{2} s \int_0^\infty \frac{du}{u^{3/2}} e^{-s^2/4u} e^{-uA^2} \qquad (87)$$

$$
\begin{aligned}
\Phi_\epsilon(E) &= H_0 - E + \mu(\epsilon) \\
&- \frac{\sqrt{\pi}}{4}\lambda^2(\epsilon) \int_0^\infty du \int_0^\infty ds\, s e^{-s^2/4} K_{u+2\epsilon}(a,a) \\
&\quad \frac{1}{\sqrt{u}(H_0 - E)} \left[1 - e^{-s\sqrt{u}(H_0-E)}\right] \\
&- \frac{\pi}{4}\lambda^2(\epsilon) \int d_g^3x\, d_g^3y \int_0^\infty ds\, s^2 \\
&\times \int_0^\infty du_1 \frac{e^{-s^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2 \frac{e^{-s^2/4u_2}}{u_2^{3/2}} \\
&\times K_{\epsilon+u_1}(a,y) K_{\epsilon+u_2}(a,x) \\
&\times \quad \phi^{(-)}(y) e^{-s(H_0-E)} \phi^{(+)}(x)
\end{aligned}
$$

$$
\frac{\mu(\epsilon)}{\lambda^2(\epsilon)} = \frac{\mu_R}{\lambda_R^2} + \frac{\pi}{2} \int_0^\infty du\, K_{u+2\epsilon}(a,a)
$$

$$
\frac{1}{\lambda^2(\epsilon)} = \frac{1}{\lambda_R^2} - \sqrt{\pi} \int_0^\infty du\, \sqrt{u} K_{u+2\epsilon}(a,a)
$$

$$
\begin{aligned}
\frac{\Phi(E)}{\lambda_R^2} \;=\; & \frac{(H_0-E)}{\lambda_R^2}+\frac{\mu_R}{\lambda_R^2} \\
& -\frac{\sqrt{\pi}}{4}\int_0^\infty du\int_0^\infty ds\, s e^{-s^2/4}\frac{K_u(a,a)}{\sqrt{u}(H_0-E)} \\
& \times[1-s\sqrt{u}(H_0-E)+\frac{1}{2}s^2u(H_0-E)^2 \\
& -e^{-s\sqrt{u}(H_0-E)}] \\
& -\frac{\pi}{4}\int d_g^3x\, d_g^3y\int_0^\infty ds\, s^2 \\
& \int_0^\infty du_1\,\frac{e^{-s^2/4u_1}}{u_1^{3/2}}\int_0^\infty du_2\,\frac{e^{-s^2/4u_2}}{u_2^{3/2}} \\
& \times K_{u_1}(a,y)K_{u_2}(a,x) \\
& \phi^{(-)}(y)e^{-s(H_0-E)}\phi^{(+)}(x)
\end{aligned}
$$

• there is a wave function renormalization!

$$
\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{88}
$$

the new one should be,

$$
\sqrt{Z(\epsilon)}\tilde{\chi}_- = \chi_- \quad \tilde{\chi}_+ = \chi_+. \tag{89}
$$

• This makes all the elements of the Resolvent finite!

♯ one should also choose $\mu_P$ such that $E = \mu_p$ becomes a zero of $\Phi$ for the vacuum $|0>$.

Then the answer in the flat case can be recast into the form:

$$\Phi(E) = (H_0 - E + \mu_P)\Big(1 + \frac{\lambda_R^2}{2}\int \frac{[d^3p]}{\omega(p)}$$
$$\times \Big[\frac{1}{(H_0 - E + \omega(p))(\omega(p) - \mu_P)} - \frac{1}{\omega^2(p)}\Big]\Big)$$
$$- \lambda_R^2 \int \frac{[d^3pd^3q]}{\sqrt{4\omega(p)\omega(q)}} a^\dagger(p) \frac{1}{H_0 - E + \omega(p) + \omega(q)} a(q)$$

♠ further problems:

*show that the Hamiltonian is bounded from below*

*show that the mean field theory gives a nice answer*

*show that the above formulae really define a quantum Hamiltonian*

- Thank you for your patience!

$$\frac{d}{ds} g(\frac{DY}{Ds}, Y) = \underbrace{-g(R(Y, \frac{d\gamma}{ds})\frac{d\gamma}{ds}, Y)}_{>0} + \underbrace{g(\frac{DY}{Ds}, \frac{DY}{Ds})}_{>0} \quad (90)$$

• geodesic deviation-Jacobi fields

$$\frac{D^2 Y}{Ds^2} + R(Y, \frac{d\gamma}{ds})\frac{d\gamma}{ds} = 0. \quad (91)$$

• sectional curvature This is a map from two planes of the tangent space to $\mathbf{R}$:

$$K : Gr_2(TM) \to \mathbf{R}$$
$$K(U, V) = \frac{g(R(U, V)V, U)}{[g(U, U)g(V, V) - g(U, V)^2]}.$$

If the sectional curvature is everywhere negative, then there are no conjugate points.

♯ Density of states, discontinuity accross the branch cut:

$$\mathcal{P}(\frac{1}{H-z+i\epsilon} - \frac{1}{H-z-i\epsilon}) \qquad (92)$$

• why heat kernel? Fundamental solution to the heat diffusion in a medium. Balance eqn + current law:

$$c\frac{\partial T}{\partial t} + \nabla . j = 0, \quad j = -\kappa \nabla T. \qquad (93)$$

• Initial temperature given $T(x, t = 0) = h(x)$

$$T(x,t) = \int d_g x \; K_t(x,y) h(y), \qquad (94)$$

solves the initial value problem.

♠ (Digression) geometric invariants:

$$\text{Tr}e^{-t\Delta_g} = \int d_g x K_t(x,x)$$
$$\approx \frac{1}{(4\pi t)^{n/2}} \sum_k \underbrace{\left(\int d_g x\, u_k(x,x)\right)}_{\text{geometric invariant}} t^k \text{ as } t \to 0^+$$

If two manifolds are isospectral then these invariants should all be the same!

$$
\begin{aligned}
\frac{\partial \Sigma^m(\nu)}{\partial \nu} &= (A^m(\nu), \frac{\partial \Phi(-\nu^2)}{\partial \nu} A^m(\nu)) \\
&= \int_0^\infty dt\ 2\nu t e^{-\nu^2 t} \sum_{i,j} K_t(a_i, a_j) A_i^{m*}(\nu) A_j^m(\nu) \\
&= \sum_\lambda |\sum_i A_i^m(\nu)\phi_\lambda(a_i)|^2 2\nu \int_0^\infty t dt e^{-(\lambda^2+\nu^2)t} \\
&= \sum_\lambda \frac{2\nu |\sum_i A_i^m(\nu)\phi_\lambda(a_i)|^2}{(\lambda^2+\nu^2)^2} > 0. \qquad (95)
\end{aligned}
$$